

Phase transitions and noise cross-correlations in a model of directed polymers in a disordered medium

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We show that effective interactions mediated by disorder between two directed polymers can be modeled as the cross-correlation of noises in the Kardar-Parisi-Zhang (KPZ) equations satisfied by the respective free energies of these polymers. When there are two polymers, disorder introduces attractive interactions between them. We analyze the phase diagram in detail and show that these interactions lead to new phases in the phase diagram. We show that, even in dimension $d=1$, the two directed polymers see the attraction only if the strength of the disorder potential exceeds a threshold value. We extend our calculations to show that if there are m polymers in the system, then m -body interactions are generated in the disorder averaged effective free energy.

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I. INTRODUCTION

A. Background

Studies of phase transitions in presence of disorder have opened up new problems in statistical mechanics. Phase transitions in spin glasses and polymers in disordered media are typical examples [1–5]. Various attempts have been made to understand such transitions see, e.g., Refs. [6,7]. In Ref. [6] replica trick has been used while in Refs. [5,7,9] dynamic renormalization group has been used and the relevant exponents (see Sec. II) have been calculated. The nature of overlaps, if there is more than one polymer in the system, has also been examined [7]. However, whether directed polymers (DP) can interact via disorder in the absence of any *direct* interaction has not yet been studied. In this paper, we ask: If there are more than one DPs in the disordered medium, can they interact through disorder in absence of any direct mutual interaction? We find that in the absence of any direct interactions, disorder mediates effective attractive interactions between the DPs that lead to binding transitions between them, which are qualitatively similar to [7].

A directed polymer in $d+1$ dimensions is just a directed string stretched along one particular direction with free fluctuations in all other d transverse directions. The Hamiltonian of a directed polymer in a quenched random potential is

$$\frac{H}{k_B T} = \int_0^t d\tau \left[\frac{\nu}{2} \left(\frac{d\mathbf{x}}{d\tau} \right)^2 + \frac{\lambda}{2\nu} V(\mathbf{x}(\tau), \tau) \right], \quad (1)$$

where $x(t)$ is the d -dimensional transverse spatial coordinate of the directed polymer at length t . The first term is the energy due to transverse fluctuations (elastic energy) and the second one is the potential energy due to disorder. In a random environment there is a competition between the potential energy due to randomness and the elastic energy. The random potential V is chosen to be Gaussian-distributed, zero mean with a correlation given by

$$\langle V(\mathbf{k}, t) V(\mathbf{k}', t') \rangle = 2D \delta^d(\mathbf{k} + \mathbf{k}') \delta(t - t'). \quad (2)$$

The elastic term $\nu(d\mathbf{x}/d\tau)^2$ attempts to smoothen out the DP [minimum $\nu(dx/d\tau)^2$ everywhere], while the disorder potential favors the DP to take a rough profile so that the DP can go through the low energy paths. At a low enough temperature the second option is energetically favorable and a disorder-dominated super diffusive phase is produced [2,5,6,10,11]. At $d>2$ a transition is observed from low temperature strong disorder phase to high temperature smooth phase [5,12,13], which is described by an unstable fixed point to $O(\epsilon)$ in a dynamic RG calculation [5,10]. There is a one-to-one mapping between the problem of a directed polymer in a random medium and the nonequilibrium surface growth problem described by the Burgers/Kardar-Parisi-Zhang (KPZ) equation. The Burgers equation is the simplest nonlinear generalization of the diffusion equation [14]. This equation is used to describe diverse phenomena: structure formation in astrophysical situations, turbulence, etc. This can be mapped into the Kardar-Parisi-Zhang (KPZ) equation, which is a prototype model for nonequilibrium growth surfaces:

$$\frac{\partial h}{\partial t} + \frac{\lambda}{2} (\nabla h)^2 = \nu \nabla^2 h + \eta. \quad (3)$$

The Gaussian noise satisfies

$$\langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = D \delta^d(\mathbf{x} - \mathbf{x}') \delta(t - t'). \quad (4)$$

The long wavelength and long time properties of this equation have been studied extensively using dynamic renormalization group method [1].

Interestingly, this equation can be transformed into a linear equation by Cole-Hopf transformation [1]:

$$h(x, t) = (2\nu/\lambda) \ln W \equiv k_B T \ln W. \quad (5)$$

The resultant linear equation is a diffusion equation with a multiplicative noise:

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$$\frac{\partial W}{\partial t} = \nu \nabla^2 W + \frac{\eta}{2\nu} W. \quad (6)$$

The partition function Z of a directed polymer satisfies the above equation with t being the coordinate parametrizing the length of the polymer. It immediately follows that the free energy of the DP, $h \equiv k_B T \ln Z$ satisfies the KPZ Eq. (3). The corresponding Hamiltonian is the Hamiltonian of a directed polymer as given in Eq. (1). The relation between the surface growth problem described by the KPZ equation and the directed polymer problem described by the Hamiltonian (1) is as follows: The temperature scale of the polymer has been set to one; hence the elastic modulus of the polymer is given by $c \equiv 1/2\nu$, and the height variable h of the KPZ equation gives the free energy of the polymer problem. The probability distribution of the quench random potential $V(\mathbf{x}, t)$ is same as that of the noise $f(\mathbf{x}, t)$ in the KPZ Eq. (3). The relevant exponents are χ and $\zeta = 1/z$. χ describes the free energy fluctuations $f \sim t^{\chi/z}$ (χ is also the roughness exponent of the height field h . z is given by $\langle [x(t) - x(0)]^2 \rangle \sim t^{2/z}$ and is also the dynamic exponent of h . Due to the Galilean invariance of the KPZ equation there is an exact exponent relation

$$\chi + z = 2. \quad (7)$$

B. Results

In this paper, we investigate when there are two DPs in a random medium whether the medium can induce interactions leading to phase transitions in absence of any direct mutual interactions between the polymers. We find that disorder indeed induces effective attractive interactions between the polymers that causes phase transitions in the system. In terms of the KPZ descriptions, this implies that noises in the individual KPZ equations satisfied by the free energies of the respective polymers have nonzero cross-correlations. In Sec. II, we show that if there are two polymers in the system, then cross-correlations of the random potentials seen by them may lead to a binding-unbinding transition of them. We analyze the phase diagram and show that new phases appear, which do not exist when there is only one polymer in the medium. We calculate the crossover exponent (defined below). In Sec. III, we generalize our calculations and show that effective m -polymer interactions are also generated. We summarize our results in Sec. IV.

II. TWO DIRECTED POLYMERS IN A RANDOM MEDIUM

A. Model

In absence of any direct interactions, the Hamiltonian of two DPs in a random medium becomes just the sum of the individual Hamiltonians H_1 and H_2 :

$$H = H_1 + H_2 = \int_0^\tau \left[\frac{\nu}{2} \left(\frac{d\mathbf{x}_1}{dt} \right)^2 + \frac{\nu}{2} \left(\frac{d\mathbf{x}_2}{dt} \right)^2 + V_1(\mathbf{x}_1, t) + V_2(\mathbf{x}_2, t) \right], \quad (8)$$

In two papers [7], Mukherjee and Bhattacharjee have studied overlap of directed polymers in a random medium (see also [8] for a related problem on a lattice). They calculated the overlap of polymers by introducing new interactions in the Hamiltonian (8) and calculated the scaling behavior using a dynamic renormalization group approach. Their modified Hamiltonian is given by [7]

$$H_m = \sum_{i=1}^m H_i + (\lambda/2\gamma) v_m \int_0^t d\tau \prod_{i=1}^{m-1} \delta([\mathbf{x}_{i,i+1}(\tau)]), \quad (9)$$

where H_i is the Hamiltonian (1) for the i th polymer and $\mathbf{x}_{i,i+1} = \mathbf{x}_i - \mathbf{x}_{i+1}$. The presence of the additional δ -functions interactions ensure that the overlap between N polymers is nonzero. Here, H_m is the Hamiltonian (9). However, due to the additional terms in their Hamiltonian representing the overlap, the equation that is satisfied by the corresponding total free energy is not quite the usual KPZ equation; it has an additional term as noise:

$$\frac{\partial h}{\partial t} = \sum_{j=1}^m \left[\gamma \nabla_j^2 h + \frac{\lambda}{2} (\nabla_j h)^2 \right] + g_0, \quad (10)$$

where

$$g_0 = \sum_{j=1}^m \left[\gamma V(x_j, t) + v_m \prod_{j=1}^{m-1} \delta(x_{j,j+1}(\tau)) \right]. \quad (11)$$

Due to this unusual looking noise term, although Eq. (10) looks like a higher (md) dimensional KPZ equation, it is not really so. With this modified Hamiltonian/KPZ equation the crossover exponents ϕ_m for overlap of m chains for $v_m \rightarrow 0$ have been calculated in Ref. [7]. We, however, do not include additional interactions and work with the Hamiltonian (8).

B. Effective interactions between two directed polymers: The phase diagram

Let us again consider the Hamiltonian for two DPs in a disordered medium:

$$H = H_1 + H_2 = \sum_{i=1,2} \int_0^t d\tau \left[\frac{\nu}{2} \left(\frac{d\mathbf{x}_i}{d\tau} \right)^2 + \frac{\lambda}{2\nu} V(\mathbf{x}_i(\tau), \tau) \right], \quad (12)$$

where H_1 and H_2 are the Hamiltonians for the two DPs, respectively, \mathbf{x}_1 and \mathbf{x}_2 are their transverse fluctuations. We take the total Hamiltonian to be additive (i.e., a sum of H_1 and H_2) as there are no direct mutual interactions between the polymers. $V_1 = V(\mathbf{x}_1)$ and $V_2 = V(\mathbf{x}_2)$ are the potential energies. The total partition function Z is the product of the individual partition functions:

$$Z = Z_1 Z_2 = \int \mathcal{D}x_1 \mathcal{D}x_2 \exp[-\beta(H_1 + H_2)].$$

Z_1 and Z_2 satisfy

$$\frac{\partial Z_1}{\partial t} = \nu \nabla^2 Z_1 + Z_1 V_1, \quad (13)$$

$$\frac{\partial Z_2}{\partial t} = \nu \nabla^2 Z_2 + Z_2 V_2, \quad (14)$$

while the free energies h_1 and h_2 satisfy [as $Z_{1,2} = \exp(-\beta h_{1,2})$]

$$\frac{\partial h_1}{\partial t} = \frac{\lambda}{2} (\nabla h_1)^2 + \nabla^2 h_1 + f_1, \quad (15)$$

$$\frac{\partial h_2}{\partial t} = \frac{\lambda}{2} (\nabla h_2)^2 + \nabla^2 h_2 + f_2. \quad (16)$$

Let us calculate [the (cross)-correlation function of the transverse fluctuations of the two polymers] $\overline{\int \mathcal{D}\mathbf{x}_1 \mathcal{D}\mathbf{x}_2 \mathbf{x}_1(t_1) \mathbf{x}_2(t_2) \exp[-\beta(H_1 + H_2)] / Z}$. An overbar indicates averaging over disorder realizations. We may evaluate it in lowest order in λ in a perturbation expansion. Expanding the Boltzmann factor in powers of λ , we obtain

$$\begin{aligned} & \overline{\int \mathcal{D}\mathbf{x}_1 \mathcal{D}\mathbf{x}_2 \mathbf{x}_1 \mathbf{x}_2 \exp[-\beta(H_1 + H_2)] / Z} \\ &= \int \Pi \mathcal{D}\mathbf{x}_i \mathbf{x}_i \exp \left[-\beta \left(\sum_{i=1,2} \frac{d\mathbf{x}_i^2}{d\tau} \right) 1 + \lambda \bar{V}_{1c} + \lambda^2 \bar{V}_{2c} \right. \\ & \quad \left. + \lambda^2 (\bar{V}_1^2/2)_c + \lambda^2 (\bar{V}_2^2/2)_c + \lambda^2 (\bar{V}_1 \bar{V}_2)_c + \dots \right], \quad (17) \end{aligned}$$

In the right-hand side a subscript c indicates cumulants, i.e., contributions only from the connected diagrams should be considered. We take

$$\begin{aligned} \langle V(\mathbf{x}_1, t) V(\mathbf{x}'_1, t') \rangle &= 2D_1 \delta(\mathbf{x}_1 - \mathbf{x}'_1) \delta(t - t'), \\ \langle V(\mathbf{x}_2, t) V(\mathbf{x}'_2, t') \rangle &= 2D_2 \delta(\mathbf{x}_2 - \mathbf{x}'_2) \delta(t - t'). \quad (18) \end{aligned}$$

These we call ‘‘autocorrelations’’ in a sense that in Eq. (18) coordinates refer to the same DP. It is clear that unless $\langle V_1(\mathbf{x}_1, t) V_2(\mathbf{x}_2, t) \rangle$ is nonzero, the cross-correlation function (as defined above) vanishes identically. We choose

$$\langle V_1(\mathbf{x}_1, t) V_2(\mathbf{x}_2, t') \rangle = 2\bar{D} \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(t - t'), \quad (19)$$

where \bar{D} may differ from D_1 or D_2 as the two polymers may be distinguishable. We want to investigate the effect of this on the phase diagram of the polymers. We give a physical interpretation of this shortly below. Let us see what this condition means in terms of the KPZ description: The free energies h_1 and h_2 of the two polymers satisfy usual KPZ equations. Without any loss of generality, we put $D_1 = D_2 = D$. We have [see Eq. (18)]

$$\langle f_1(k, t) f_1(k', t') \rangle = 2D \delta^d(k + k') \delta(t - t'), \quad (20)$$

$$\langle f_2(k, t) f_2(k', t') \rangle = 2D \delta^d(k + k') \delta(t - t'). \quad (21)$$

Such a choice, with $\langle f_1 f_2 \rangle = 0$ would automatically guarantee that h_1 and h_2 represent the free energies of two *identical, mutually noninteracting* polymers in a disordered medium. It is clear that having a nonzero $\langle V_1(k, t) V_2(k', t') \rangle$ implies a nonzero $\langle f_1(k, t) f_2(k', t') \rangle$:



FIG. 1. A schematic diagram showing an effective contact interaction between two polymers that arises due to cross-correlation of the random potential (see text).

$$\langle f_1(k, t) f_2(k', t') \rangle = 2\bar{D} \delta^d(k + k') \delta(t - t'). \quad (22)$$

If $\langle f_1(x, t) f_2(x', t') \rangle = 0$ or $\langle V_1(x, t) V_2(x', t') \rangle = 0$, then after averaging over disorder the effective Hamiltonian is just the sum of two noninteracting single chain Hamiltonians. Obviously, there is no attractive interaction between the two polymers. However, since both the polymers are in the same random medium, there are of course correlations between the random potentials seen by the two polymers, which, as we will see, mediate interactions between the polymers. Equivalently said, the polymers interact through the disordered medium. Mathematically, on averaging over the disorder distribution, due to the nonzero cross-correlations of the noises, a new term $\propto \bar{D} \delta^d[x_1(\tau) - x_2(\tau)]$ is generated in the effective Hamiltonian. This new ‘‘effective potential’’ can be interpreted as an attractive interaction felt by one of the polymers when it is in contact with the other (see Fig. 1). This will cause, as we have seen before, quantities like $\langle \mathbf{x}_1 \mathbf{x}_2 \rangle$ to be nonzero for certain strengths of the interactions. We show below that this effective attractive interaction leads to a phase transition involving the two DPs.

The total free energy F of the two DPs after averaging over disorder distribution is a function of the couplings \bar{D} : $F \equiv F(\bar{D}, t)$. We define, as in Ref. [7] the order parameter as the derivative of the quenched free energy with respect to the appropriate coupling constant:

$$q(t) = \frac{1}{t} \int_0^t d\tau \delta(x_1(\tau) - x_2(\tau)) = - \frac{1}{t} \frac{dF(\bar{D}, t)}{d\bar{D}} \Big|_{\bar{D}=0}. \quad (23)$$

Here $F(\bar{D}, t)$ is the scaling part of the effective free energy for the two polymers. Following Ref. [7] we start with a scaling form

$$F(\bar{D}, t) = t^{\chi/z} f(\bar{D} t^{-\phi_2/z}). \quad (24)$$

Here, ϕ_2 is the crossover exponent. This gives [7]

$$q = t^{\Sigma_2} Q(D t^{-\phi_2/z}), \quad (25)$$

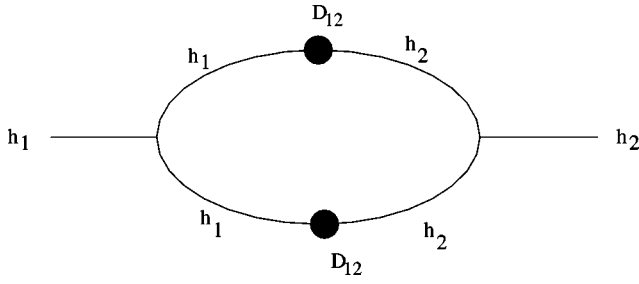


FIG. 2. The one-loop diagram, contributing to the renormalization of \tilde{D} ; continuous lines indicate h_1 or h_2 lines as mentioned in the figure and the small filled circles indicate bare $\tilde{D}=D_{12}$.

where $\Sigma_2=(\chi-\phi_2-z)/z$. We calculate the crossover exponent in a one-loop dynamic renormalization group calculation.

Renormalization group flow equations for the parameters of a single KPZ equation (namely, for ν , D , and λ) have already been calculated [1]. We present the calculation for \tilde{D} : The following diagram will contribute at the one-loop level (Fig. 2).

The one-loop integral, after frequency integration becomes $\tilde{D}^2 \int [d^d q / (2\pi)^d] (1/\nu^3 q^2) \sim \Lambda^{d-2}$, where Λ is some momentum scale coming from the lower limit of the one-loop integral. We see that this integral has the same infrared behavior as the usual one-loop integral that comes in for the renormalization of the noise correlations in a single KPZ equation. Under rescaling of space and time, different parameters scale according to their naïve dimensions: $\nu \rightarrow b^{z-2}\nu$, $\lambda \rightarrow b^{z+\chi-2}\lambda$, $D \rightarrow b^{z-d-2\chi}D$, and $\tilde{D} \rightarrow b^{z-d-2\chi}\tilde{D}$. The flow equations for different coupling constants are

$$\frac{d\nu}{dl} = [z-2+k_d g(2-d)/4d]\nu, \quad (26)$$

$$\frac{d\lambda}{dl} = [z+\chi-2]\lambda, \quad (27)$$

$$\frac{dD}{dl} = [z-d-2\chi+gk_d/4]D, \quad (28)$$

$$\frac{d\tilde{D}}{dl} = [z-d-2\chi+\tilde{g}k_d/4]\tilde{D}, \quad (29)$$

where a is the smallest length scale in the problem, and $k_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(d/2)$ and $g=(a/\pi)^{2-d}D\lambda^2/\nu^3$ and $\tilde{g}=(a/\pi)^{2-d}\tilde{D}\lambda^2/\nu^3$ are the dimensionless coupling constants. It is easy to obtain the flow equations for the coupling constants:

$$\frac{dg}{dl} = (2-d)g + k_d \frac{2d-3}{2d} g^2, \quad (30)$$

$$\frac{d\tilde{g}}{dl} = \tilde{g} \left(2-d + k_d \frac{\tilde{g}}{4} - 3k_d g \frac{2-d}{4d} \right). \quad (31)$$

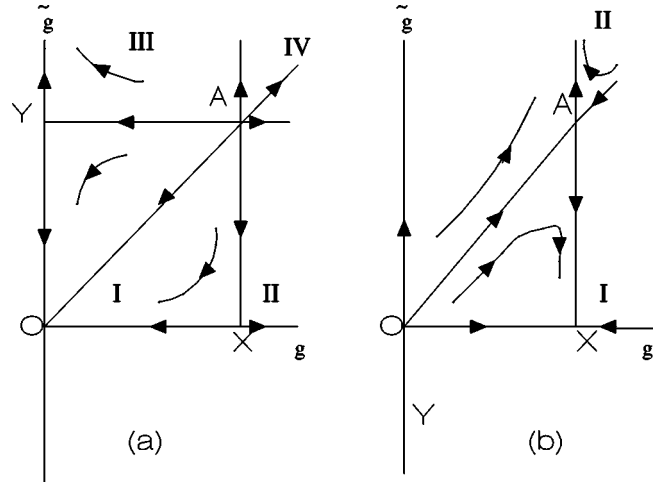


FIG. 3. A diagram indicating the flow around the fixed points in $g-\tilde{g}$ plane: (a) $d=2+\epsilon$, (b) $d=1$.

These equations have fixed point solutions O , [$g_0=0, \tilde{g}=0$]; X , [$g=2d(d-2)/k_d(2d-3) \equiv g_c, \tilde{g}=0$], Y , [$g=0, \tilde{g}=d-2$], A , [$g=g_c, \tilde{g}=g_c$]. We show the flows in a fixed point diagram in Fig. 3.

Among these values, $O:(0,0)$ for $d=2+\epsilon$ are stable and correspond to Gaussian polymers. At $d=1$, $g=g_c$, and $\tilde{g}=\tilde{g}_c$ are the stable fixed points implying that any small amount of disorder makes the polymer nonGaussian. At $d=2+\epsilon$, $Y \equiv (g=0, \tilde{g}=d-2)$, $X \equiv (g=g_c, \tilde{g}=0)$, $A \equiv (g=g_c, \tilde{g}=g_c)$ are all unstable. A is unstable along both the directions, X is unstable along g direction, and Y is unstable along \tilde{g} direction. Hence they indicate second-order phase transitions. At $d=1$, A is still unstable along \tilde{g} direction. Y takes a negative ordinate, reflecting presumably a bound state not describable by a fixed point. So if one moves along AX one always encounters a second-order phase transition. The disorder averaged Hamiltonian has potential terms of the forms $-g\delta(\mathbf{x}_1-\mathbf{x}'_1)$, $-g\delta(\mathbf{x}_2-\mathbf{x}'_2)$, $-\tilde{g}\delta(\mathbf{x}_1-\mathbf{x}_2)$. The last term comes from the cross-correlation effects. Due to the negative sign it is attractive in nature. Recall that in $d=1$ X is stable along g direction indicating that any small amount of disorder is relevant in $d=1$. However, X is unstable along \tilde{g} direction, which shows that unless the disorder strength exceeds a minimum value, the two polymers do not attract each other. We obtain the following phases in the phase diagram (see Fig. 3) characterized by stable fixed points.

(1) In Fig. 3(a), i.e., for $d=2+\epsilon$ we have the following.

(a) Free Gaussian polymers characterized by the stable fixed point $O(0,0)$ (denoted by I).

(b) Free “strong-disorder-KPZ” polymers, i.e., polymers do not attract each other but their individual fluctuations are influenced by the disorder (denoted by II). This phase is characterized by a stable fixed point of the form $(G_1,0)$, not accessible in perturbation theory.

(c) Bound Gaussian polymers: individual polymer fluctuations are unaffected by disorder but they are bound due to the effective attractive interaction (denoted by III). This phase is characterized by a stable fixed point of the form $(0,\tilde{G}_1)$, not accessible in perturbation theory.

(d) Bound KPZ polymers: individual polymer fluctuations are affected by disorder, also they form a bound pair due to an effective attractive interaction (denoted by IV). This phase is characterized by a stable fixed point of the (G, \tilde{G}) again not accessible in perturbation theory.

(2) In Fig. 3(b), i.e., for $d=1$, we have the following.

(a) Free ‘‘strong-disorder/KPZ’’ polymers described by stable fixed point (2,0) (denoted by I). Exponents are known *exactly*.

(b) Bound KPZ polymers (denoted by II) by a stable fixed point not accessible in perturbation theory.

Note that in Fig. 3, the line $g = \tilde{g}$ is invariant under the RG transformation; this is just a reflection of the fact that if the two DPs are indistinguishable, they remain so in a coarse grained description.

In Ref. [7] overlaps have been calculated at $(g = g_c, \tilde{g} = g_c)$. We, however, investigate the nature of the phase transitions considering \tilde{g} as the ordering field at $g = \epsilon$ and $g = 0$. Hence our definitions of order parameter are given by

$$q \sim \left. \frac{dF}{dD^*} \right|_{D^*=0}, \quad (32)$$

where D^* is an effective coupling constant that is a function of \tilde{g} . In other words, we wish to calculate the crossover exponent ϕ_2 for the two polymers at the fixed point $(g_c, 0)$ in the g - \tilde{g} plane. It is given by

$$\phi_2 = 2\chi + d - z + \delta_{12} = 0. \quad (33)$$

Hence, we obtain $\Sigma_2 = (\chi - z)/z$ in all dimensions. In particular, in $d=1$, $\Sigma_2 = -1/2$ and in $d=2 + \epsilon$, $\Sigma_2 = -1$ [since $\chi = 0$ in $O(\epsilon)$]. The length scale exponent ν at the unstable fixed points A and Y in Fig. 3(a) (i.e., for $d > 2$) is the same, however at O in Fig. 3(b) (i.e., $d=1$) it is different from A.

III. EFFECTIVE INTERACTIONS INVOLVING ARBITRARY NUMBER OF POLYMERS

In this section, we generalize our previous calculations to show that when there are m DPs the effective free energy can have any arbitrary n -polymer ($n \leq m$) interactions. We show that these interaction terms naturally arise in the free energy after disorder averaging. Let us examine the quantity $\int \prod_{i=1, m} \mathcal{D}\mathbf{x}_i \exp[-\beta H]$, where $H = H_1 + \dots + H_m$ is the sum of the Hamiltonians of m polymers [each being the same as Eq. (1)]. Following our calculations of the previous section, we write

$$\begin{aligned} & \int \prod_i \mathcal{D}\mathbf{x}_i \exp(-\beta H) / Z \\ &= \int \prod \mathcal{D}\mathbf{x}_i \exp\left(\sum_i \frac{d\mathbf{x}_i^2}{d\tau}\right) \\ & \quad \times [1 + \dots + \lambda^m (V_1, \dots, V_m)]. \end{aligned} \quad (34)$$

It is clear that the m -point cross-correlation function of transverse fluctuations is nonzero only if $\langle V_1, \dots, V_m \rangle_c$ is nonzero. We calculate $\langle V_1, \dots, V_m \rangle$ to $O(\lambda^m)$ (\equiv one-loop) in the KPZ description of the problem. As we shall see later,

$$\begin{aligned} \langle V_1(\mathbf{k}_1, t_1) \dots V_m(\mathbf{k}_m, t_m) \rangle &\propto D_m \delta(\mathbf{k}_1 + \dots + \mathbf{k}_m) \\ &\quad \times \delta(t_1 - t_m) \dots \delta(t_{m-1} - t_m). \end{aligned} \quad (35)$$

We find out the scaling dimension of D_m in a one-loop approximation. We examine below the nature of the transition due to D_m ; an effective interaction that makes many point correlation functions $\langle x_1, \dots, x_m \rangle$ nonvanishing. Now the effective free energy (averaged over random potential) of the m polymers is a function of all these effective couplings:

$$F \equiv F(\tilde{D}, \dots, D_m). \quad (36)$$

Following the usual definition of order parameter as the derivative of the free energy with respect to the appropriate coupling constant, we obtain [7]

$$q_m = - \left. \frac{1}{t} \frac{dF(D_m t^{-\phi_m/z})}{dD_m} \right|_{D_m=0}. \quad (37)$$

It is easy to see that, in our approach, each of the m polymers' free energy will satisfy the usual KPZ equation separately. As in Sec. II, we assume noises present in these m KPZ equations will have nonzero cross-correlations between them:

$$\frac{\partial}{\partial t} h_1 + \frac{\lambda}{2} (\nabla h_1)^2 = \nu \nabla^2 h_1 + f_1, \quad (38)$$

$$\frac{\partial}{\partial t} h_2 + \frac{\lambda}{2} (\nabla h_2)^2 = \nu \nabla^2 h_2 + f_2, \quad (39)$$

...

$$\frac{\partial}{\partial t} h_m + \frac{\lambda}{2} (\nabla h_m)^2 = \nu \nabla^2 h_m + f_m, \quad (40)$$

with noise correlations given as in Sec. II:

$$\langle f_i(k, t) f_j(k', t') \rangle = D \delta(k + k') \delta(\omega + \omega'), \quad (41)$$

$$\langle f_i(k, t) f_j(k', t') \rangle = \tilde{D} \delta(k + k') \delta(\omega + \omega'); i \neq j. \quad (42)$$

We define the relevant order parameter as

$$q_m = - \frac{1}{t} \int_0^t d\tau \left\langle \prod_{i=1}^{m-1} \delta(x_i(\tau) - x_m(\tau)) \right\rangle. \quad (43)$$

We note that even though our definition of order parameter is actually different from the one defined in Ref. [7], physically it measures the same thing. For $m=2$, the two definitions are identical. It is evident that $\langle V_1, \dots, V_m \rangle \propto \langle f_1, \dots, f_m \rangle$.

A. Effective interactions between three polymers

Let us consider a situation where we have three polymers. Due to the Gaussian statistics of the random potential, $\langle h_1(\mathbf{x}_1, t) h_2(\mathbf{x}_2, t) h_3(\mathbf{x}_3, t) \rangle$ is zero at the bare level. However, due to the nonlinearity in the equation, $\langle h_1(\mathbf{x}_1, t) h_2(\mathbf{x}_2, t) h_3(\mathbf{x}_3, t) \rangle$ is nonzero at the one-loop level. In Fig. 4 we show the one-loop diagrams.

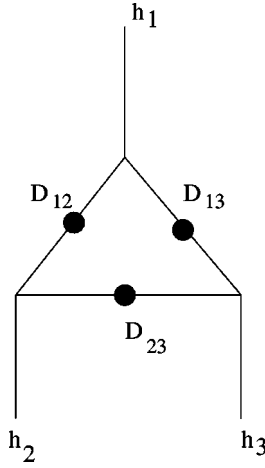


FIG. 4. A diagram showing (a) one-loop contribution to $\langle h_1 h_2 h_3 \rangle$. The filled circles refer to $D_{12}=D_{13}=D_{23}=\tilde{D}$.

The one-loop diagram in Fig. 4 scales as

$$\sim \tilde{D}^3 \lambda^3 \int \frac{d^d q}{\nu^5 q^4} \sim \tilde{D}^3 \lambda^3 \Lambda^{d-4} \sim g_c^{5/3} \Lambda^{d-4}, \quad (44)$$

where Λ is the lower cutoff of the momentum integral. We see that even though the bare noise is Gaussian the effective noise has nonzero third cumulant (which arises due to the nonlinear term in the KPZ equation):

$$\begin{aligned} \langle f_1(\mathbf{k}_1, t_1) f_2(\mathbf{k}_2, t_2) f_3(\mathbf{k}_3, t_3) \rangle \\ = D_3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \delta(t_1 - t_2) \delta(t_1 - t_3). \end{aligned} \quad (45)$$

The naive dimension of D_3 is $z - 2d - 3\chi$ and anomalous dimension is $4 - d$ [see Eq. (44)]. Hence

$$\phi_3 = -(z - 2d - 3\chi + 4 - d) = -(4 - 3d) = 3\epsilon \quad (46)$$

at $d = 2 + \epsilon$ and [since $z = 2 + O(\epsilon^2)$ and $\chi = O(\epsilon^2)$] consequently,

$$\Sigma_3 = (\chi - \phi_3 - z)/z = (-3\epsilon - 2)/2 \quad (47)$$

at $d = 2 + \epsilon$. We notice that $\Sigma_3 < 0$, indicating that in the thermodynamic limit i.e., when $t \rightarrow \infty$, $q_3 \sim t^{\Sigma_3} \sim |T - T_c|^{-\nu \Sigma} \rightarrow 0$ as $T \rightarrow T_c^-$. Here, ν is the correlation length exponent.

B. Effective interactions between four polymers

Let us consider a situation when we have four polymers in the medium. We define the order parameter

$$q_4 \equiv -\frac{1}{t} \int_0^t d\tau \langle \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(\mathbf{x}_1 - \mathbf{x}_3) \delta(\mathbf{x}_1 - \mathbf{x}_4) \rangle. \quad (48)$$

Similar to our analysis of three polymers, we calculate $\langle f_1(\mathbf{x}_1, t_1) f_2(\mathbf{x}_2, t_2) f_3(\mathbf{x}_3, t_3) f_4(\mathbf{x}_4, t_4) \rangle$ in a one-loop perturbation theory. The one-loop diagram is shown in Fig. 5. The one-loop integrals scale as $\sim (\tilde{D}^4/\nu^7) \Lambda^{d-6} \sim \epsilon^{7/3} \Lambda^{d-6} \sim D_4$ and disappear if \tilde{D} vanishes. As in our previous analysis, this can be interpreted as if the free energies h_1, \dots, h_4 satisfy a linear equation and the noises have four-point cross-correlations given by

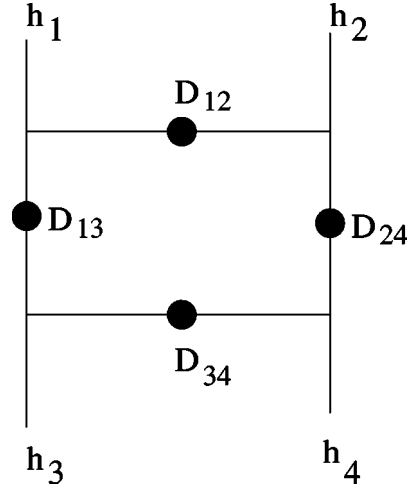


FIG. 5. A schematic one-loop diagram contributing to $\langle f_1 f_2 f_3 f_4 \rangle$. $D_{12}=D_{23}=D_{34}=D_{14}=\tilde{D}$.

$$\langle f_1 f_2 f_3 f_4 \rangle \equiv D_4 \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(\mathbf{x}_1 - \mathbf{x}_3) \delta(\mathbf{x}_1 - \mathbf{x}_4). \quad (49)$$

We calculate q_4 from the relation

$$q_4 = -\frac{1}{t} \frac{dF(D_4 t^{-\phi_4/z})}{D_4} \Big|_{D_4=0} \sim t^{\Sigma_4}. \quad (50)$$

We find $\phi_4 = -(z - 3d - 4\chi + 6 - d) = 4\epsilon$. Hence, we obtain $\Sigma_4 = (-4\epsilon - 2)/2$, i.e., $q_4 \sim t^{(-3\epsilon - 2)/2}$. Figure 5 shows a one-loop diagram contributing to $\langle f_1 f_2 f_3 f_4 \rangle$, $D_{12}=D_{23}=D_{34}=D_{14}=\tilde{D}$.

C. Effective interactions between m polymers

It is easy to convince oneself that any higher point correlation of the random potential is nonzero. Notice that these m -body effective interactions are built from the pairwise cross-correlations. Since the two-chain effective interaction is attractive, these m -chain interactions constructed out of that are also attractive. Figure 6 is a typical diagram that contributes at $O(\lambda^m)$ (\equiv one-loop) to $\langle V_1, \dots, V_m \rangle$ or to $\langle f_1, \dots, f_m \rangle$. A nonzero value of this ensures phase transi-

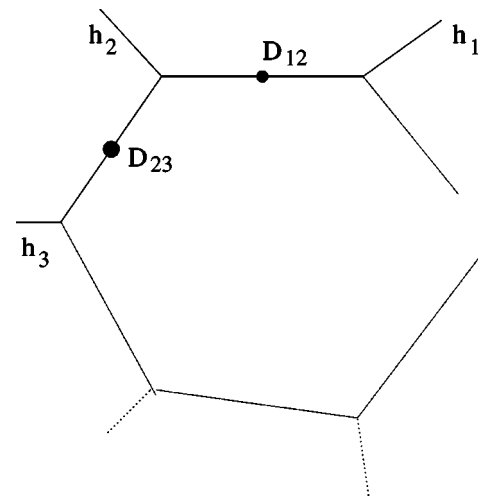


FIG. 6. Diagrammatic contribution to $\langle f_1, f_2, \dots, f_m \rangle$ up to one-loop order.

tions involving m polymers. The one-loop integrals scale as $\sim \Lambda^{\epsilon+4-2m}$. We immediately obtain

$$\phi_m = -[z - (m-1)d - m\chi + 2m - 4 - \epsilon] = m\epsilon. \quad (51)$$

This leads to

$$q_m \sim t^{\Sigma_m}, \quad \Sigma_m = [-(m)\epsilon - 2]/2, \quad (52)$$

at $d = 2 + \epsilon$. Naturally, the effective free energy contains all the interaction terms that are generated due to disorder averaging. Equivalently, the disorder mediates interactions between arbitrary numbers of polymers.

At $d = 1$, along AX , $\phi_m = -1/2 - m/2 < 0$ ($m > 2$), hence the effective m -point coupling disappears in the large length scale limit. Thus there is now m -body interaction in that limit. However, $\phi_2 = 0$, i.e., effective two-body attractive interaction is marginal, causing two polymers to attract.

IV. TRANSITIONS ALONG THE LINE YO

So far we focused on transitions at $g \sim \epsilon$, i.e., along the line AX . But one could have followed a different path in the $(g-\tilde{g})$ plane: In particular, if one follows the line $g = 0$ then the unstable fixed point $(0, \tilde{g}^*)$ gives rise to transitions with different exponents. Along this line $g^* = 0$, hence *individual* polymers are free (i.e., randomness of the medium is irrelevant as far as their transverse fluctuations are concerned; these fluctuations are still described by $z = 2$ and $\chi = d/2$ in d dimensions). However these free polymers still see attractive contact interactions in presence of one another that cause these transitions. At dimension $d = 2 + \epsilon$, $g^* = 0$, $z = 2$, and $\chi = d/2$; hence, at this fixed point:

$$\phi_m = -[z - (m-1)d - m\chi + 2m - 4 - \epsilon] = \left(\frac{3}{2}m - 1\right)\epsilon, \quad (53)$$

$$\Sigma_m = (\chi - \phi_m - z)/z = \left[\frac{3}{2}m\epsilon - 2\right] / 2. \quad (54)$$

Physically along this line polymers are individually free (described by Gaussian polymer exponents), no matter what the value of \tilde{g} is. If the value of \tilde{g} is higher than a critical value then Gaussian polymers attract each other due to contact interactions induced by disorder between them.

V. SUMMARY

In this paper, we have discussed how disorder generates attractive interactions between two directed polymers in a disordered medium. We show that due to these attractive interactions the polymers undergo a binding-unbinding transition. We argue that in terms of the KPZ language this is due to the cross-correlations of noises in the two KPZ equations satisfied by the respective free energies of the two polymers. We present a detailed analysis of the phase diagram. The strength of the cross correlation (i.e., the strength of the effective attractive interaction) appears as a new parameter in the problem. This is relevant, in an RG sense, above a threshold value in any space dimension d . In $d = 2 + \epsilon$, we get a new nontrivial unstable fixed point signaling a second-order binding-unbinding transition. At $d = 2 + \epsilon$, we get four different phases, two of which are new arising due to the cross-correlation only. Similarly in $1d$ two phases appear, one of them is new. These effects can be realized by putting different kinds of polymers in the medium. Notice that if the two DPs are identical then $g = \tilde{g}$ and there is no independent variation of g or \tilde{g} . In that case, the system is described by the line $g = \tilde{g}$. It will be very interesting to examine this issue of disorder-induced phase transitions of DPs by using a variational replica approach [15] or a replica Bethe ansatz approach [16]. In a typical replica calculation, the n -replica Hamiltonian H_n becomes a function of the replica-replica interaction terms. In the present case, in a replica calculation, H_n will be the replica Hamiltonian for $2n$ DPs. It will involve ‘‘cross-replica’’ interaction terms arising due to the cross-correlation of the random potential. It will be interesting to see how the results obtained in this paper can be obtained in a replica approach.

Note added in proof. The problem of two identical polymers with attractive interactions was solved for arbitrary dimension d in Ref. [17]. The author wishes to thank M. Lässig for pointing this out.

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